Playing games in quantum mechanical settings: A necessary and sufficient condition

Junichi Shimamura,[1] Şahin Kaya Özdemir,[1,*] and Nobuyuki Imoto[1]

[1]*SORST Research Team for Interacting Carrier Electronics, CREST Research Team for Photonic Quantum Information, Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan*

A number of recent studies [1, 2, 3, 4, 5, 6, 7, 8, 9] have focused on novel features in game theory when the games are played using quantum mechanical toolbox (entanglement, unitary operators, measurement). Researchers have concentrated in two-player-two strategy, 2×2, dilemma containing classical games, and transferred them into quantum realm showing that in quantum pure strategies dilemmas in such games can be resolved if entanglement is distributed between the players armed with quantum operations. Moreover, it became clear that the players receive the highest sum of payoffs available in the game, which are otherwise impossible in classical pure strategies. Encouraged by the observation of rich dynamics of physical systems with many interacting parties and the power of entanglement in quantum versions of 2×2 games, it became generally accepted that quantum versions can be easily extended to N-player situations by simply allowing N-partite entangled states. In this article, however, we show that this is not generally true because the reproducibility of classical tasks in quantum domain imposes limitations on the type of entanglement and quantum operators. We propose a benchmark for the evaluation of quantum and classical versions of games, and derive the necessary and sufficient conditions for a physical realization. We give examples of entangled states that can and cannot be used, and the characteristics of quantum operators used as strategies.

Mathematical models and techniques of game theory have increasingly been used by computer and information scientists, i.e., distributed computing, cryptography, watermarking and information hiding tasks can be modelled as games [10, 11, 12, 13, 14, 15, 16]. Therefore, new directions have been opened in the interpretation and use of game theoretical toolbox which has been traditionally limited in economical and evolutionary biology problems [17]. This is not a surprise because there is a very strong connection between the two: Information [18]. Game theory deals with situations where players make decisions, and then depending on their decisions, the outcome of games are determined. This process can be modelled as the flow of information. Since generation, transmission, storage, manipulation and processing of information need physical means, information is governed by the laws of physics. Therefore, information is closely related to physics and hence to quantum mechanics. In short, information is the common link among game theory, physics, quantum mechanics, computation and information sciences. Along this line of thinking, researchers introduced the quantum mechanical toolbox into game theory to see what new features will arise combining these two beautiful areas of science.

Quantum mechanics is introduced into game theory through the use of quantum bits (qubits) instead of classical bits, entanglement which is a quantum correlation with a highly complex structure and is considered to be the essential ingredient to exploit the potential power of quantum information processing, and the quantum operations. This effort, although has been criticized on the basis of using artificial models [19, 20], has produced significant results: (i) Dilemmas in some games can be resolved [1, 7, 8, 21, 22, 23], (ii) playing quantum games can be more efficient in terms of communication cost; less information needs to be exchanged in order to play the quantized versions of classical games [8, 9, 18], and (iii) entanglement is not necessary for the emergence of Nash Equilibrium but for obtaining the highest possible sum of payoffs [8], and (iv) Quantum advantage does not survive in the presence of noise above critical level [8, 24]. In addition to these fundamental results and efforts, Piotrowski and Sladkowski have described, in a series of papers, market phenomena, bargaining, auction and finance using quantum game theory [25, 26, 27]. The positive results are consequences of the fact that sharing entanglement, using quantum operators and measurements allow players to have a greater number of strategies to choose from when compared to the situation in classical games.

In this paper, we focus on (i) the extent of entangled states and quantum operators that can and cannot be used in multi-player games, and (ii) comparison of the results of classical games and their quantized versions on a fair basis by introducing a benchmark. Moreover, this study attempts to clarify a relatively unexplored area of interest in quantum game theory, that is the effects of different types of entangled states and their use in multi-player multi-strategy games in quantum settings. Our approach, which will become clear in the following, to these points are based on the reproducibility of classical games in the physical schemes used for the implementation of their quantized versions.

We should mention that reproducibility requires that a chosen model of game should simulate both quantum and classical versions of the game to make a comparative analysis of quantum and classical strategies, and to discuss what can or cannot be attained by introducing quantum mechanical toolbox. This is indeed what has been observed in quantum Turing machine (QTM). A QTM can simulate the classical Turing machine (CTM) under special conditions, and can reproduce the results of the original CTM. Therefore, we think the reproducibility criterion must be taken into consideration whenever a comparison between classical and quantum versions of a task is needed. An important consequence of this criterion in game theory is the main contribution of this study: The derivation of the necessary and sufficient condition for entangled states and quantum operators that can be used in the quantized versions of classical games.

Definitions and models: We start by introducing some basic definitions and the model of the quantized classical game that is considered in this study. In classical game theory, a strategic game is defined by $\Gamma = [N, (S_i)_{i \in N}, (\$_i)_{i \in N}]$ where N is the set of players, $S_i = \{s_i^1, s_i^2, \ldots, s_i^m\}$ is the set of

pure strategies available to the i-th player with m being the number of strategies, and $\$_i$ is the payoff function for the i-th player. When the strategic game Γ is played with pure strategies, the i-th player chooses one of the strategies from the set S_i. With all players applying a pure strategy (each player chooses only one strategy from the strategy set), the joint strategy of the players is denoted by $\vec{s}_k = (s_1^{l_1}, s_2^{l_2}, \ldots, s_N^{l_N})$ with $l_i \in \{1, 2, 3, \ldots, m\}$ and $k = \sum_{i=1}^{N}(l_i - 1)m^{i-1}$. Then the i-th player's payoff function is represented by $\$_i(\vec{s}_k)$ when the joint strategy set $\vec{s}_k$ is chosen, i.e., the payoff functions of all players corresponding to the unique joint strategy $\vec{s}_k$ can be represented by $\vec{\$} = (\$_1(\vec{s}_k), \$_2(\vec{s}_k), \ldots, \$_N(\vec{s}_k))$ and it is uniquely determined from the payoff matrix of the game. Players may choose to play with mixed strategies, that is they randomize among their pure strategies resulting in the expected payoff

$$F_i(q_1, \cdots, q_N) = \sum_{s_1^{l_1} \in S_1} \cdots \sum_{s_N^{l_N} \in S_N} \left\{ \prod_{j=1}^{N} q_j(s_j^{l_j}) \right\} f_i(s_1^{l_1}, \cdots, s_N^{l_N}) \tag{1}$$

where $q_j(s_j^{l_j})$ represents the probability that j-th player chooses the pure strategy $s_j^{l_j}$ and f_i is the corresponding payoff function for the i-th player.

Most of the studies on quantum versions of classical games have been based on the model proposed by Eisert *et al.* [1]. In this model, the strategy set of the players consists of unitary operators which are applied locally on a shared entangled state by the players. A measurement by a referee on the final state after the application of the operators maps the chosen strategies of the players to their payoff functions. In this model, the two strategies of the players in the original classical game is represented by two unitary operators, $\{\hat{\sigma}_0, i\hat{\sigma}_y\}$, i.e., in Prisoner's Dilemma $\{\hat{\sigma}_0$ and $i\hat{\sigma}_y\}$ respectively corresponds to "Cooperate" and "Defect".

In this study, however, we consider the following model of a quantum version of classical games for N-player-two-strategy games, which is more general than Eisert *et al.*'s model [1] and includes it. In our model [29], (i) A referee prepares an N-qubit entangled state $|\Psi\rangle$ and distributes it among N players, one qubit for each player. (In order to see features intrinsic to quantumness, we focus on a shared entangled state among the players, and exclude the trivial case where a product state is distributed.) (ii) Each player independently and locally applies a unitary operator chosen from the SU(2) set on his qubit, i.e, the i-th player applies $\hat{u}_i$. (We restrict ourselves to the entire set of SU(2) because the global phase is irrelevant). Hence, the combined strategies of all the players is represented by the tensor product of all players' unitary operators as $\hat{x} = \hat{u}_1 \otimes \hat{u}_2 \otimes \cdots \otimes \hat{u}_N$, which generates the output state $\hat{x}|\Psi\rangle$ to be submitted to the referee. (iii) Upon receiving this final state, the referee makes a projective measurement $\{\Pi_j\}_{j=1}^{2^N}$ which outputs j with probability $\mathrm{Tr}[\Pi_j \hat{x}|\Psi\rangle\langle\Psi|\hat{x}^\dagger]$, and assigns payoffs chosen from the payoff matrix, depending on the measurement outcome j. Therefore, the expected payoff of the i-th player is described by

$$F_i(\hat{U}_1, \cdots, \hat{U}_N) = \mathrm{Tr}\left[\left(\sum_j a_j^i \Pi_j\right)\left(\hat{U}_1 \otimes \cdots \otimes \hat{U}_N |\Psi_{\mathrm{in}}\rangle\langle\Psi_{\mathrm{in}}|\hat{U}_1^\dagger \otimes \cdots \otimes \hat{U}_N^\dagger\right)\right] \tag{2}$$

where Π_j is the projector and a_j^i is the i-th player's payoff when the measurement outcome is j. This model can be implemented in a physical scheme with the current level of experimental techniques and technology of quantum mechanics.

Reproducibility criterion: In the following, reproducibility criterion corresponds to the reproducibility of a multi-player two-strategy classical game in the quantization model explained above. This criterion requires that the expected payoff given in eq. (1) is reproduced in the quantum version, too [29].

First, we consider the reproducibility problem only in pure strategies, $q_j(s_j^{l_j}) = 1$ for all $j = 1 \cdots N$, (conditions for the reproducibility including the mixed strategies will be discussed later below). In this model, we require that a classical game be reproduced when each player's strategy set is restricted to two unitary operators, $\{\hat{u}_i^1, \hat{u}_i^2\}$, corresponding to the two pure strategies in the classical game. When the classical game is played in this model, the combined pure strategy of the players is represented by $\hat{x}_k = \hat{u}_1^{l_1} \otimes \hat{u}_2^{l_2} \otimes \cdots \otimes \hat{u}_N^{l_N}$ with $l_i = \{1, 2\}$ and $k = \sum_{i=1}^{N}(l_i - 1)2^{i-1}$. Thus the output state becomes $|\Phi_k\rangle = \hat{x}_k|\Psi\rangle$ with $k = \{1, 2, \ldots, 2^N\}$. When the strategy combination $\hat{x}_k$ is played, Eq. (2) becomes

$$\begin{aligned} F_i(\hat{u}_1^{l_1}, \ldots, \hat{u}_N^{l_N}) &= \mathrm{Tr}\left[\left(\sum_j a_j^i \Pi_j\right)\hat{x}_k|\Psi\rangle\langle\Psi|\hat{x}_k^\dagger\right] \\ &= \sum_j a_j^i \mathrm{Tr}[\Pi_j \hat{x}_k|\Psi\rangle\langle\Psi|\hat{x}_k^\dagger] \end{aligned} \tag{3}$$

where the measurement outcome j occurs with probability $\mathrm{Tr}[\Pi_j(\hat{x}_k|\Psi\rangle\langle\Psi|\hat{x}_k^\dagger]$. Playing with pure strategies requires referee discriminate all the possible output states $|\Phi_k\rangle$ deterministically in order to assign payoffs uniquely. That is, the projector $\{\Pi_j\}_{j=1}^{2^N}$ has to satisfy $\mathrm{Tr}[\Pi_j|\Phi_k\rangle\langle\Phi_k|] = \delta_{jk}$, which is possible if and only if

$$\langle\Phi_\alpha|\Phi_\beta\rangle = \delta_{\alpha\beta} \quad \forall\alpha, \beta. \tag{4}$$

Under this distinguishability condition, we see that $F_i(\hat{u}_1^{l_1}, \ldots, \hat{u}_N^{l_N}) = a_k^i = f_i(s_{l_1}, \ldots, s_{l_N})$. Therefore, Eq. (4) becomes the *necessary condition* for the reproducibility of classical games in the quantum model. Imposing this necessary condition on several multiparty-entangled states, we have found [29]: (a) Bell states and any two-qubit pure state satisfy it if the two unitary operators for two players are chosen as $\{\hat{\sigma}_0, \hat{\sigma}_x\}$ and $\{\hat{\sigma}_0, i\hat{\sigma}_y\}$, respectively. (b) Multipartite GHZ-like states of the form $(|00\ldots0\rangle + i|11\ldots1\rangle)/\sqrt{2}$ satisfy the above condition if the unitary operators of the players are chosen as $\{\hat{\sigma}_0, i\hat{\sigma}_y\}$. Entangled states that can be obtained from GHZ state by local unitary transformations also satisfy it. (c) N-party form of the W state, defined as $|W_N\rangle = |N-1, 1\rangle/\sqrt{N}$ $(N \geq 3)$, where $|N-1, 1\rangle$ is a symmetric state with $N-1$ zeros and 1 one, e.g. $|2, 1\rangle = |001\rangle + |010\rangle + |100\rangle$, does not satisfy it, therefore the entangled state $|W_N\rangle$ cannot be used in this model of quantum games. (d) Among the Dicke states, which is a class of symmetric states represented as $|N-m, m\rangle/\sqrt{{}_N C_m}$ with $(N-m)$ zeros and m ones (${}_N C_m$ denoting the binomial coefficient), only the states $|1, 1\rangle$ and $|2, 2\rangle$ satisfy the distinguishability condition.

Quantum operators and distinguishability condition: In the following, we will discuss some basic properties of quantum operators which satisfy the distinguishability

condition, and show that this condition is also the sufficient condition for the reproducibility of classical games.

Let us assume that for a given entangled state, $|\Psi\rangle$, satisfying the distinguishability condition, we find two unitary operators corresponding to the classical pure strategies as required in the model proposed above for each player. Moreover, considering that only the first player changes his operator while the others stick to their first operator, we obtain $|\Phi_0\rangle = \hat{u}_1^1 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1 |\Psi\rangle$ and $|\Phi_1\rangle = \hat{u}_1^2 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1 |\Psi\rangle$. Imposing the distinguishability criterion on this simple case, we arrive at the condition,

$$\langle \Psi | (\hat{u}_1^1)^\dagger \hat{u}_1^2 \otimes \hat{I} \otimes \cdots \otimes \hat{I} | \Psi \rangle = 0. \tag{5}$$

Since $(\hat{u}_1^1)^\dagger \hat{u}_1^2$ is a normal operator, it can be diagonalized by a unitary operator $\hat{z}_1$. Furthermore, since $(\hat{u}_1^1)^\dagger \hat{u}_1^2$ is a SU(2) operator, the eigenvalues are given by $e^{i\phi_1}$ and $e^{-i\phi_1}$. Then, we can transform eq. (5) into

$$\begin{aligned} &\langle \Psi | \hat{z}_1^\dagger \left(\hat{z}_1 (\hat{u}_1^1)^\dagger \hat{u}_1^2 \hat{z}_1^\dagger \right) \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} | \Psi \rangle \\ &\quad = \langle \Psi' | \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{bmatrix} \otimes \hat{I} \otimes \cdots \otimes \hat{I} | \Psi' \rangle = 0, \end{aligned} \tag{6}$$

where $|\Psi'\rangle = \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\Psi\rangle$. We write the state $|\Psi'\rangle$ on computational basis as $|\Psi'\rangle = \sum_{i_j \in \{0,1\}} c_{i_1 i_2 \ldots i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle$ and substitute this into Eq. (6). After some straightforward matrix and trigonometric manipulations we obtain

$$\begin{aligned} &\langle \Psi' | \begin{bmatrix} e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_1} \end{bmatrix} \otimes \hat{I} \otimes \cdots \otimes \hat{I} | \Psi' \rangle \\ &\quad = e^{i\phi_1} \sum_{i_j \in \{0,1\}} |c_{0\, i_2 \ldots i_N}|^2 + e^{-i\phi_1} \sum_{i_j \in \{0,1\}} |c_{1\, i_2 \ldots i_N}|^2 \\ &\quad = \cos\phi_1 + i \left(2 \sum_{i_j \in \{0,1\}} |c_{0\, i_2 \ldots i_N}|^2 - 1 \right) \sin\phi_1 = 0. \end{aligned} \tag{7}$$

In order for the above equality to hold, $\cos\phi_1 = 0$ and $2\sum_{i_j \in \{0,1\}} |c_{0\, i_2 \ldots i_N}|^2 - 1 = 0$ must be satisfied. The equation $\cos\phi_1 = 0$ implies that the diagonalized form $\hat{D}_1 = \hat{z}_1 (\hat{u}_1^1)^\dagger \hat{u}_1^2 \hat{z}_1^\dagger$ can be written as

$$\hat{D}_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{8}$$

This argument holds for all players, therefore we write $\hat{z}_k (\hat{u}_k^1)^\dagger \hat{u}_k^2 \hat{z}_k^\dagger = \hat{D}_k = i\hat{\sigma}_z$ for $k = 1, \cdots, N$. For example, let us consider the case of the four-party Dickie State $|2,2\rangle$, which satisfies the distinguishability criterion with the unitary operators $\hat{u}_{k=1,2,3,4}^1 = \hat{I}$, $\hat{u}_{k=1,2,3}^2 = i(\sqrt{2}\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{3}$, and $\hat{u}_4^2 = i\hat{\sigma}_y$. It can easily be verified that the eigenvalues of ${\hat{u_k}}^{1\dagger} \hat{u}_k^2$ for all players are i and $-i$, and they are already in the diagonalized form of Eq. (8). For the GHZ state, all players should have the operators $\hat{u}_k^1 = \hat{I}$ and $\hat{u}_k^1 = i\hat{\sigma}_y$ is the same as the operator set of the fourth player of $|2,2\rangle$, and therefore can be written as in Eq. (8).

Next we consider the following scenario: Each player has two operators satisfying the above properties. Instead of choosing either of these operators, they prefer to use a linear combination of their operator set. Let this operator be $\hat{w}_k = \cos\theta_k \hat{u}_k^1 + \sin\theta_k \hat{u}_k^2$ for the k-th player. Then, we ask the questions (i) Does the property of the operators $\hat{u}_k^1$ and $\hat{u}_k^2$ derived from the distinguishability condition impose any condition on the operator $\hat{w}_k$?, and (ii) What does the outcome of the game played in the quantum version with the operator $\hat{w}_k$ imply? Since $\hat{z}_k (\hat{u}_k 1)^\dagger \hat{u}_k 2 \hat{z}_k^\dagger$ is in the diagonalized form $\hat{D}$, we can write $\hat{w}_k$ in such a way that it contains $\hat{D}$. In order to do this, we look at the operator $\hat{w}_k^\dagger \hat{w}_k$ which is given as

$$\begin{aligned} \hat{w}_k^\dagger \hat{w}_k &= (\hat{u}_k^{1\dagger} \cos\theta_k + \hat{u}_k^{2\dagger} \sin\theta_k)(\hat{u}_k^1 \cos\theta_k + \hat{u}_k^2 \sin\theta_k) \\ &= \hat{I} + \cos\theta_k \sin\theta_k (\hat{u}_k^{1\dagger} \hat{u}_k^2 + \hat{u}_k^{2\dagger} \hat{u}_k^1) \\ &= \hat{I} + \cos\theta_k \sin\theta_k (\hat{z}_k^\dagger \hat{D} \hat{z}_k + \hat{z}_k^\dagger \hat{D}^\dagger \hat{z}_k) \\ &= \hat{I} + \cos\theta_k \sin\theta_k (\hat{z}_k^\dagger \hat{D} \hat{z}_k - \hat{z}_k^\dagger \hat{D} \hat{z}_k) \\ &= \hat{I}, \end{aligned} \tag{9}$$

where we have used $\hat{u}_k^{1\dagger} \hat{u}_k^2 = \hat{z}_k^\dagger \hat{D} \hat{z}_k$, and $\hat{D}^\dagger = -\hat{D}$ since $\hat{D}$ is anti-hermitian. Therefore, as seen in Eq. (9), the distinguishability condition requires that $\hat{w}_k$ be a unitary operator. In order to find the outcome of the game when players use the operators $\hat{w}_k = \cos\theta_k \hat{u}_k^1 + \sin\theta_k \hat{u}_k^2$, we substitute $\hat{w}_k$ into Eq. (2) and obtain

$$\begin{aligned} F_k(\hat{w}_1, \ldots, \hat{w}_N) &= a_1^k \cos^2\theta_1 \cos^2\theta_2 \cdot\cdot \cos^2\theta_N + a_2^k \sin^2\theta_1 \cos^2\theta_2 \cdot\cdot \cos^2\theta_N + \cdots + a_{2^N}^k \cos^2\theta_1 \sin^2\theta_2 \cdot\cdot \sin^2\theta_N \\ &= a_1^k p_1 p_2 \cdot\cdot p_N + a_2^k (1-p_1) p_2 \cdot\cdot p_N + \cdots + a_{2^N}^k (1-p_1)(1-p_2) \cdot\cdot (1-p_N) \\ &= \sum_{s_1 \in S_1} \cdot\cdot \sum_{s_N \in S_N} \left(\prod q_1(s_1) \ldots q_N(s_N) \right) f_k(s_1, .., s_N), \end{aligned} \tag{10}$$

where $p_k = \cos 2\theta_k$ represents the probability that k-th player chooses the strategy $s_{1_k} \in S_k$. We can see that Eq. (10) has the same form of the expected payoff given in Eq. (1) for the classical game implying that when players choose $\hat{w}_k = \cos\theta_k \hat{u}_k + i \sin\theta_k \hat{v}_k$ as their strategies, the payoff for the mixed strategies in the classical games is reproduced in this quantum version. Therefore, we conclude that the distinguishability condition of Eq. (4) is the *necessary and sufficient condition* for the reproducibility of a classical game in the quantum version. This is because when players apply

their pure strategies with unit probabilities, results of classical pure strategy, and when they apply a linear combination of their pure strategies results of classical mixed strategy are reproduced in the quantum setting.

Entangled states and distinguishability condition: After stating the properties of operators which satisfy the distinguishability criterion, we proceed to investigate the properties of the class of entangled states which satisfy the distinguishability condition.

Suppose that an N-qubit state $|\,\Psi\rangle$ and two unitary operators $\{\hat{u}_k, \hat{v}_k\}$ satisfy the distinguishability criterion. Let us consider the distinguishability criterion between the states $|\,\Phi_0\rangle = \hat{u}_1 \otimes \hat{u}_2 \otimes \cdots \otimes \hat{u}_N |\,\Psi\rangle$ and $|\,\Phi_1\rangle = \hat{v}_1 \otimes \hat{u}_2 \otimes \cdots \otimes \hat{u}_N |\,\Psi\rangle$. Using the properties of the operators derived above, the distinguishability criterion for these two states is written as

$$\begin{aligned} &\langle \Psi\,|\hat{z}_1^\dagger\, \hat{z}_1 \hat{u}_1^\dagger \hat{v}_1 \hat{z}_1^\dagger\, \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi\rangle \\ &\qquad = \langle \Psi'\,|\hat{D}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi'\rangle = 0, \end{aligned} \tag{11}$$

where $\hat{z}_1$ is a unitary operator diagonalizing $\hat{u}_1^\dagger \hat{v}_1$ and $|\,\Psi'\rangle = \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi\rangle$. This implies that if the N-qubit state $|\,\Psi\rangle$ and the operators $\{\hat{u}_k, \hat{v}_k\}$ satisfy the distinguishability criterion, then the state $|\,\Psi'\rangle = \hat{z}_1 \otimes \hat{z}_2 \otimes \cdots \otimes \hat{z}_N |\,\Psi\rangle$ and the unitary operators $\{\hat{D}, \hat{I}\}$ should satisfy the distinguishability criterion, too. Since the global phase is irrelevant, Eq. (11) can be further reduced to

$$\langle \Psi'\,| \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi'\rangle = 0. \tag{12}$$

Thus, we end up with the following $2^N - 1$ equalities to be satisfied for the distinguishability criterion:

$$\begin{aligned} &\langle \Psi'\,|\hat{\sigma}_z \otimes \hat{I} \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi'\rangle = 0, \\ &\langle \Psi'\,|\hat{I} \otimes \hat{\sigma}_z \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi'\rangle = 0, \\ &\qquad \vdots \\ &\langle \Psi'\,|\hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \cdots \otimes \hat{\sigma}_z |\,\Psi'\rangle = 0. \end{aligned} \tag{13}$$

When $|\,\Psi'\rangle$ is described as $\sum_{i_j \in \{0,1\}} c_{i_1 i_2 \ldots i_N} |\,i_1\rangle |\,i_2\rangle \cdots |\,i_N\rangle$, these equations and the normalization condition can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & \ldots & -1 & -1 \\ & & \ldots & & \\ & & \vdots & & \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \begin{bmatrix} |c_{00\ldots 0}|^2 \\ |c_{00\ldots 1}|^2 \\ \vdots \\ |c_{11\ldots 1}|^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{14}$$

where the last row is the normalization condition. The row vector corresponds to the diagonal elements of $\hat{\sigma}_z^{\{0,1\}} \otimes \cdots \otimes \hat{\sigma}_z^{\{0,1\}}$ where $\hat{\sigma}_z^0$ is defined as $\hat{I}$. Here, let us consider the operator $\hat{x}, \hat{y} \in (\hat{\sigma}_z^{\{0,1\}})^{\otimes N}$. The product of two operators $\hat{x}\,\hat{y}$ also belongs to $(\hat{\sigma}_z^{\{0,1\}})^{\otimes N}$. Since $\mathrm{Tr}[\hat{\sigma}_z] = 0$, when $\hat{x} \neq \hat{y}$, $\mathrm{Tr}[\hat{x}\,\hat{y}] = \mathrm{Tr}[\hat{x}]\,\mathrm{Tr}[\hat{y}] = 0$. This means that every two row vectors are orthogonal with each other, thus the matrix in Eq. (14) has an inverse and $|c_{i_1 i_2 \ldots i_N}|^2$ are determined uniquely. Since each row but the last contains equal number of 1 and -1, we can easily find that $|c_{i_1 i_2 \ldots i_N}|^2 = 1/N$. This implies that if a state satisfies the distinguishability condition, then it should be transformed by local unitary operators into the state which contains all possible terms with the same magnitude but different relative phases, i.e.,

$$|\,\Psi'\rangle = \sum_{i_j \in \{0,1\}} \frac{1}{\sqrt{N}} e^{i\phi_{i_1 i_2 \ldots i_N}} |\,i_1\rangle |\,i_2\rangle \cdots |\,i_N\rangle. \tag{15}$$

As examples for this case, let us consider the product state and the GHZ state, which satisfy the distinguishability criterion. We can easily find that the product state is transformed into the form of the above state by Hadamard operator, written as $(\hat{\sigma_x} + \hat{\sigma_z})/\sqrt{2}$. The GHZ state is also transformed into the form of Eq. (15) by $(e^{i\frac{\pi}{4}} \hat{I} + e^{-i\frac{\pi}{4}} \hat{\sigma}_z + \hat{\sigma}_y)/\sqrt{2}$ for one player and Hadamard operator for the others.

Reproducibility criterion as a benchmark: This criterion requires that in any physical model to play the quantum version, it should be possible to play the classical game as well. It is only when this is possible we can compare the outcomes of classical and quantum versions to draw conclusions on whether the quantum version has advantage over the classical one or not. Therefore, the fist thing the physical scheme should provide is the availability of unitary operators corresponding to classical pure strategies. If there exists such operators then one can compare the outcomes for the pure strategies. To make this point clear, let us consider the entangled state W for which one cannot find two unitary operators $\{\hat{u}_k, \hat{v}_k\}$ satisfying the croterion. When any game is played using this entangled state with unitary operators chosen from the SU(2) set one cannot obtain the outcome of the classical game in pure strategies. Moreover, the payoffs that will be obtained become a probability distribution over the entries of the payoff matrix of the classical game. Therefore, comparing the quantum version using W state with the classical game in pure strategies is not fair. In the same way, comparing the quantum version played with GHZ and W states is not fair either because in GHZ the payoffs delivered to the players are unique entries from the classical payoff table for GHZ state, thus original classical game results in pure strategies are reproduced which is not the case for W state. Therefore, we think that the reproducibility criterion constitutes a benchmark not only for the evaluation of entangled states and operators in quantum games but also in the comparative analysis of classical games and their corresponding quantum versions.

Conclusion: In this paper, for the first time, we give the necessary and sufficient condition to play quantized version of classical games in a physical scheme. This condition is introduced here as the "reproducibility criterion," or "distinguishability condition" and it provides a fair basis to compare quantum versions of games with their classical counterparts. This benchmark requires that results of the classical games be reproduced in the model of the quantum version. The necessary and sufficient condition we give here shows that a large class of multipartite entangled states cannot be used in the quantum version of classical games; and the operators that might be used should have a special form in their diagonalized form. Given two unitary operators $\{\hat{u}_k, \hat{v}_k\}$ corresponding to classical pure strategies and satisfying the distinguishability criterion, we can reproduce the results of classical games in pure strategies in the physical scheme. Moreover, provided that the players choose unitary operators in the space spanned by $\hat{u}_k$ and $\hat{v}_k$, mixed strategy results of classical games can be also reproduced in the quantum setting.

The authors thank M. Koashi, F. Morikoshi and T. Yamamoto for helpful discussions and warm support during this research.

* Electronic address: ozdemir@qi.mp.es.osaka-u.ac.jp

[1] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83**, 3077 (1999).
[2] D. A. Meyer, Phys. Rev. Lett. **82**, 1052 (1999).
[3] L. Marinatto, and T. Weber, Phys. Lett. A **272**, 291 (2000).
[4] S. C. Benjamin and P. M. Hayden, Phys. Rev. A **64**, 030301 (2001).
[5] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou, and R. Han, Phys. Rev. Lett. **88**, 137902 (2002).
[6] A. P. Flitney, and D. Abbott, Proc. R. Soc. (London) A **459** 2463-74 (2003).
[7] J. Shimamura, S. K. Ozdemir, F. Morikoshi and N. Imoto, Int. J. of Quant. Inf., **2/1**, 79 (2004).
[8] S. K. Ozdemir, J. Shimamura, F. Morikoshi, and N. Imoto, Phys. Lett. A, **333**, 218 (2004).
[9] C. F. Lee, N. F. Johnson, Phys. World **15(10)**, 25-29 (2002).
[10] P Moulin, and A. Ivanovic, Proc. of Int. Conf. on Image Process., **3**, 975 - 978 (2001).
[11] A. S. Cohen, A. Lapidoth, IEEE Trans. on Inf. Theory, **48**, 1639 - 1667 (2002).
[12] J. Conway, N. Sloane, IEEE Trans. on Inf. Theory, **32**, 337 - 348, (1986).
[13] X. M. Shen, L. Deng, IEEE Trans. on Signal Process., **45**, 1092 - 1095 (1997).
[14] J. M. Ettinger, 2nd Inf. Hiding Workshop, Portland, OR, Apr 15-17, (1998).
[15] S. Pateux, G. Le Guelvouit, Elsevier, Signal Process.: Image Comm., **18**, 283-296, (2003).
[16] L. A. DaSilva, and V. Srivastava, The First Workshop on Games and Emergent Behaviors in Distributed Computing Environments, Birmingham, UK, September, (2004)
[17] E. Rasmusen, *Games and Information: An Introduction to Game Theory* (Blackwell Pub, Oxford, 2001), R. B. Myerson, *Game Theory: Analysis of Conflict* (Harvard Univ. Press, Cambridge MA, 1997).
[18] C. F. Lee, and N. Johnson, Phys. Rev. A **67**, 022311 (2003).
[19] S. C. Benjamin and P. M. Hayden, Phys. Rev. Lett. **87**, 069801 (2001).
[20] S. J. van Enk and R. Pike, Phys. Rev. A **66**, 024306 (2002).
[21] J. Eisert, and M. Wilkens, J. Mod. Opt. **47**, 2543 (2000).
[22] S. C. Benjamin, P. M. Hayden, Phys. Rev. A **64(3)**,030301 (2001).
[23] A. P. Flitney, J. Ng,and D. Abbott, Physica A, **314** 35-42, (2002).
[24] N. F. Johnson, Phys. Rev. A **63**, 020302 (2001).
[25] E. W. Piotrowski, J. Sladkowski, Quant. Finance **4**, 1-7 (2004).
[26] E. W. Piotrowski, J. Sladkowski, Physica A, **312** 208 (2002).
[27] E. W. Piotrowski, J. Sladkowski, Physica A, bf 318, 505 (2003).
[28] S. K. Ozdemir, J. Shimamura, F. Morikoshi, and N. Imoto, Phys. Lett. A, **325**, 104 (2004).
[29] J. Shimamura, S. K. Ozdemir, F. Morikoshi and N. Imoto, Phys. Lett. A, **328**, 20 (2004).